

IV. "On Linear Differential Equations."—No. VII.

By W. H. L. RUSSELL, F.R.S. Received October 24, 1872.

I am desirous to conclude this series of papers with some remarks on the solutions of differential equations considered as transcendents. I shall take the linear differential equation of the second order,

$$(a + \beta x + \gamma x^2) \frac{d^2 y}{dx^2} + (a' + \beta' x + \gamma' x^2) \frac{dy}{dx} + (a'' + \beta'' x + \gamma'' x^2) y = 0,$$

which will be sufficient, as it will be seen at once that similar investigations apply generally.

Let

$$y = u_0 + u_1 x + u_2 x^2 + \dots + u_n x^n + \dots$$

We propose to investigate the general law of the convergence of this series. The general term of the series is given by the equation

$$\begin{aligned} (n+2)(n+1)au_{n+2} + (n+1)n\beta u_{n+1} + n(n-1)\gamma u_n \\ + (n+1)a'u_{n+1} + n\beta'u_n + (n-1)\gamma'u_{n-1} \\ + a''u_n + \beta''u_{n-1} + \gamma''u_{n-2} = 0. \end{aligned}$$

As (n) increases without limit, this equation becomes

$$(n+2)(n+1)au_{n+2} + (n+1)n\beta u_{n+1} + n(n-1)\gamma u_n = 0,$$

or more simply,

$$au_{n+2} + \beta u_{n+1} + \gamma u_n = 0.$$

This may be written

$$a \frac{u_{n+2}}{u_{n+1}} + \beta \frac{u_{n+1}}{u_n} + \gamma = 0.$$

Now let $\frac{u_{n+1}}{u_n}$ converge, as (n) increases without limit, to a certain quantity ρ , then $\frac{u_{n+2}}{u_{n+1}}$ will also converge toward the same quantity ρ , and ρ will be given by the equation

$$a\rho^2 + \beta\rho + \gamma = 0.$$

Let ρ_1, ρ_2 be the roots of this equation, ρ_1 the greatest root; then the series $u_0 + u_1 x + u_2 x^2 + \dots$ will be convergent if $\frac{u_{n+1}x}{u_n}$ is in the limit less than unity, or if x is less than $\frac{1}{\rho_1}$. For large values of (x) we must proceed as follows:—

Let

$$y = u_0 + \frac{u_1}{x} + \frac{u_2}{x^2} + \dots + \frac{u_n}{x^n} + \dots$$

Then, substituting in the differential equation,

$$\begin{aligned} (n-2)(n-1)au_{n-2} + n(n-1)\beta u_{n-1} + n(n+1)\gamma u_n \\ - (n-1)a'u_{n-1} - n\beta'u_n - (n+1)\gamma'u_{n+1} \\ + a''u_n + \beta''u_{n+1} + \gamma''u_{n+2} = 0. \end{aligned}$$

As (n) increases without limit, this equation may be written

$$\alpha u_{n-2} + \beta u_{n-1} + \gamma u_n = 0,$$

or

$$\alpha + \beta \cdot \frac{u_{n-1}}{u_{n-2}} + \gamma \cdot \frac{u_{n-1}}{u_{n-2}} \cdot \frac{u_n}{u_{n-1}} = 0.$$

Let $\frac{u_n}{u_{n-1}}$ converge to a certain quantity ρ , then ρ will be given by the equation

$$\alpha + \beta\rho + \gamma\rho^2 = 0.$$

Let ρ_1 be the greatest root of this equation; then the series will be convergent if $\frac{u_n}{u_{n-1}x}$ is less than unity, or if x be greater than ρ_1 .

It appears from my last paper that in certain cases a quantity of the form

$$\sqrt[n]{a+bx+\sqrt{c+ex}}$$

may be regarded as a solution of a differential equation of the form here considered. Hence we are able, by means of the principles given in this paper, to represent, under these circumstances,

$$\int dx \sqrt[n]{a+bx+\sqrt{c+ex}}$$

by a converging series when (x) does not exceed an ascertainable value. This method evidently admits of a great variety of applications.

November 30, 1872.

ANNIVERSARY MEETING.

Sir GEORGE BIDDELL AIRY, K.C.B., President, in the Chair.

Mr. Abel, for the Auditors of the Treasurer's Accounts on the part of the Society, reported that the total receipts during the past year, including a balance of £28 2s. 2d. carried from the preceding year, amount to £4308 14s. 8d., and that the total expenditure in the same period amounts to £3860 17s. 10d., leaving a balance at the Bankers of £424 0s. 3d., and £23 16s. 7d. in the hands of the Treasurer.

The thanks of the Society were voted to the Treasurer and Auditors.